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International Journal of Heat and Mass Transfer 48 (2005) 2260–2265

International Journal of HEAT and MASS TRANSFER

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Finite element solution of radiative transfer across a slab with variable spatial refractive index

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Received 12 July 2004; received in revised form 16 December 2004
Available online 14 March 2005

Abstract

To avoid the complicated computation of ray trajectories, a finite element formulation is developed to solve the radiative transfer problem in a one-dimensional absorbing–emitting–scattering semitransparent slab with variable spatial refractive index. A problem of radiative equilibrium is taken as an example to verify this finite element formulation. The predicted temperature distributions are determined by the proposed method and compared with the data in references. The results show that the finite element formulation presented in this paper has good accuracy in solving the radiative transfer in one-dimensional absorbing–emitting–scattering semitransparent medium with variable spatial refractive index.

Keywords: Radiative heat transfer; Variable spatial refractive index; Semitransparent medium; Finite element method

1. Introduction

Radiative heat transfer in semitransparent medium with variable spatial refractive index is of great interest in thermo-optical systems, and has evoked the wide interest of many researchers. As early as 1993, Siegel and Spuckler [1] analyzed the variable refractive index effects on radiation in semitransparent scattering multilayered regions, and pointed out that refractive indices of semitransparent sublayers inside a composite could have considerable effects on the temperature distribution and radiative flux fields. Due to the variable spatial refractive index, the ray goes along a curved path determined by the Fermat principle. Ray-tracing is the main difficulty for the solution of radiative transfer in the medium with variable spatial refractive index. Recently, a lot of ray-tracing techniques have been presented to solve the radiative transfer in semitransparent medium with graded refractive index. Ben Abdallah and Le Dez [2–5] developed and used a curved ray-tracing technique to analyze the radiative heat transfer in absorbing–emitting semitransparent medium with variable spatial refractive index. Huang et al. [6,7] and Xia et al. [8] presented a combined curved ray-tracing and pseudo-source adding method for radiative heat transfer in one-dimensional semitransparent medium with graded refractive index. Liu [9] developed a discrete curved ray-tracing method, in which the curved ray trajectory is locally treated as a straight line. Based on the discrete curved ray-tracing technique, Liu [10] developed a Monte Carlo discrete curved ray-tracing method, in which the Monte Carlo method is combined with the

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0017-9310/$ - see front matter
doi:10.1016/j.ijheatmasstransfer.2004.12.045

Nomenclature

b	asymmetry factor of scattering	κ_a	absorption coefficient
F	in-scattering term	κ_s	scattering coefficient
I	radiative intensity	μ	direction cosine
I_b	blackbody radiative intensity	σ	Stefan–Boltzmann constant
L	thickness of slab	τ_L	optical thickness of slab, $\tau_L = (\kappa_a + \kappa_s)L$
N	total number of unknown radiative intensity	φ	shape function
n	refractive index	Φ	scattering phase function
s	abscissa on the ray trajectory	Ψ	dimensionless radiative heat flux
T	temperature	ω	single scattering albedo, $\omega = \kappa_s/(\kappa_a + \kappa_s)$
x	axis coordinate		
		Subscripts	
Greek symbols		0	at position of $x = 0$
γ	parameter defined in Eq. (6)	L	at position of $x = L$
ε	emissivity	i, j, l	for the node i, j, or l

discrete curved ray-tracing method. Because the ray goes along a curved path determined by the Fermat principle, the ray tracing is very difficult and complex in the medium with variable spatial refractive index. Therefore, all of these methods stated above need the complicated computation of ray trajectories.

To avoid the complicated computation of ray trajectories, a method not based on curved ray-tracing needs to be developed. In this paper, we develop a finite element formulation of radiative transfer in one-dimensional absorbing–emitting–scattering semitransparent slab with variable spatial refractive index. A problem of radiative equilibrium is taken as an example to verify this finite element formulation.

2. Mathematical formulation

As shown in Fig. 1, we consider one–dimensional semitransparent gray absorbing–emitting–scattering slab with thickness L. The boundaries are opaque, diffuse and gray walls. The emissivities of boundary walls are ε_0 and ε_L, and the temperatures of boundary walls are imposed as T_0 and T_L, respectively. The absorption coefficient κ_a and scattering coefficient κ_s are uniform over the slab, but the refractive index n of medium varies with the axis coordinate x. The radiative transfer equation at steady state in an absorbing–emitting–scattering medium with spatial variable optical constants is given by

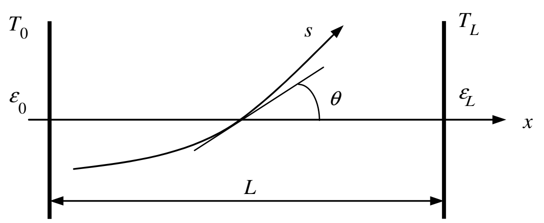

Fig. 1. Physical geometry of slab.

$$n^2 \frac{d}{ds}\left(\frac{I}{n^2}\right) + (\kappa_a + \kappa_s) I = n^2 \kappa_a I_b + \frac{\kappa_s}{2} \int_{-1}^{1} I(x, \mu') \Phi(\mu, \mu') d\mu' \tag{1a}$$

with boundary conditions

$$I(0, \mu) = \varepsilon_0 n_0^2 \frac{\sigma T_0^4}{\pi} + 2(1 - \varepsilon_0) \int_{-1}^{0} I(0, \mu') \mu' d\mu', \quad \mu \geqslant 0, \tag{1b}$$

$$I(L, \mu) = \varepsilon_L n_L^2 \frac{\sigma T_L^4}{\pi} + 2(1 - \varepsilon_L) \int_{0}^{1} I(L, \mu') \mu' d\mu', \quad \mu \leqslant 0, \tag{1c}$$

where $I(x, \mu)$ is the radiative intensity, s is the abscissa on the ray trajectory and determined by ray equation, σ is Stefan–Boltzmann constant, $\mu = \cos\theta$ is the direction cosine of the local tangent vector of ray trajectory.

In order to use the finite element method, Eq. (1a) needs to be transformed and expressed in the space of $x - \mu$. When moving along one given path, the streaming operator d/ds can be split into [11]

$$\frac{d}{ds} = \frac{\partial}{\partial x} \times \frac{dx}{ds} + \frac{\partial}{\partial \mu} \times \frac{d\mu}{ds}. \tag{2}$$

From Snell's law

$$\frac{d}{ds}(n \sin\theta) = 0, \tag{3}$$

it comes out that

$$\mu \frac{d}{dx}(n\sqrt{1 - \mu^2}) = 0, \tag{4a}$$

and

$$\frac{d\mu}{ds} = \mu\frac{d\mu}{dx} = [1-\mu^2]\frac{d\ln n}{dx}. \tag{4b}$$

Finally, by using Eqs. (2) and (4) the radiative transfer equation can be rewritten as

$$\mu\frac{\partial I}{\partial x} + \gamma(1-\mu^2)\frac{\partial I}{\partial \mu} + (\kappa_a + \kappa_s - 2\gamma\mu)I = n^2\kappa_a I_b + \frac{\kappa_s}{2}\int_{-1}^{1} I(x,\mu')\Phi(\mu,\mu')d\mu', \tag{5}$$

where

$$\gamma = \frac{d\ln n}{dx}. \tag{6}$$

It is noted that radiative intensity I is the function of x and μ. In the two-dimensional space of $x-\mu$, the domain of interest is subdivided into many elements. By using the shape function, an approximate solution of I can be assumed in the form

$$I = \sum_l I_l\varphi_l, \tag{7}$$

where the I_l is the values at the node l, and φ_l is the shape function. The weighted residuals approach is used to spatially discretize the radiative transfer equations [Eq. (5)]. Taking shape function φ_l as the weight function, Eq. (5) is weighted over the domain of interest and its integrated residuals are set to zero

$$R_l = \int_V \left[\mu\frac{\partial I}{\partial x} + \gamma(1-\mu^2)\frac{\partial I}{\partial \mu} + (\kappa_a+\kappa_s-2\gamma\mu)I - n^2\kappa_a I_b - \frac{\kappa_s}{2}\int_{-1}^{1} I(x,\mu')\Phi(\mu,\mu')d\mu'\right]\varphi_l dV = 0, \quad l = 1,2,\cdots,N, \tag{8}$$

where N is the total number of unknown radiative intensities.

In this paper, the linear triangular element is used. Each element has three nodes numbered anticlockwise as i, j and k. The weighted residuals for the element e can be written as

$$R_l^e = K_{li}^e I_i + K_{lj}^e I_j + K_{lk}^e I_k - f_l^e, \quad l = i,j,k, \tag{9a}$$

where

$$K_{lm}^e = \int_{V_e}\left[\mu\frac{\partial\varphi_m}{\partial x} + \gamma(1-\mu^2)\frac{\partial\varphi_m}{\partial\mu} + (\kappa_a+\kappa_s-2\gamma\mu)\varphi_m\right] \times \varphi_l dV, \quad l,m = i,j,k, \tag{9b}$$

$$f_l^e = \int_{V_e}\left[n^2\kappa_a I_b + \frac{\kappa_s}{2}\int_{-1}^{1} I(x,\mu')\Phi(\mu,\mu')d\mu'\right]\varphi_l dV, \quad l = i,j,k. \tag{9c}$$

In Eq. (9c), the integration of in-scattering term is evaluated approximately by the last iterated values of radiative intensity, and can be expressed as

$$G_l^e = \int_{V_e}\left[\frac{\kappa_s}{2}\int_{-1}^{1} I(x,\mu')\Phi(\mu,\mu')d\mu'\right]\varphi_l dV = \int_{V_e}[F(x_i,\mu_i)\varphi_i + F(x_j,\mu_j)\varphi_j + F(x_k,\mu_k)\varphi_k]\varphi_l dV, \quad l = i,j,k, \tag{10}$$

where

$$F(x,\mu) = \frac{\kappa_s}{2}\int_{-1}^{1} I(x,\mu')\Phi(\mu,\mu')d\mu'. \tag{11}$$

In order to compute the values of Eqs. (9) and (10), the typical 7-point numerical quadrature formula [12] in triangular elements is used. Finally, by summing the contributions from each element, the matrix system of integrated residual equation (eq. (8)) can be symbolically written as

$$K_{ij}I_j = f_i, \; i, \quad j = 1,2,\cdots,N. \tag{12}$$

Because the in-scattering term in Eq. (11) is computed approximately by using the last iterated values of radiative intensity, global iterations are necessary to include the in-scattering term and boundary conditions. From the equation derivation shown above, it can be seen that the complicated computation of ray trajectories is avoided for the finite element simulation of radiative transfer in one-dimensional absorbing–emitting–scattering semitransparent slab with variable spatial refractive index.

3. Results and discussions

To verify this finite element formulation presented above for radiative transfer in the medium with variable spatial refractive index, a problem of radiative equilibrium is taken as an example. As shown in Fig. 1, the temperatures of boundary walls are imposed as $T_0 = 1000$ K and $T_L = 1500$ K, respectively. Three particular test cases are examined. Those particular test cases are selected because exact, or at lease very precise, solutions of the radiative transfer equation exist for comparison with the finite element solution. A computer code based on the preceding calculation procedure was written. Grid refinement studies were also performed for the physical model to ensure that the essential physics are independent of grid size. For the following numerical study, the regions of $x \in [0, L]$ and $\mu \in [-1, 1]$ are divided uniformly into 40 parts, respectively, and then the two-dimensional domain of computation is subdivided into 3200 triangular elements. At radiative equilibrium, the temperature distribution within the medium is determined by

$$T(x) = \left[\frac{\pi}{2\sigma n^2(x)} \sum_{m=1}^{40} I(x, \mu_m)\Delta\mu\right]^{0.25}. \tag{13}$$

Because the blackbody radiative intensity in Eq. (9c) depends on the temperature of medium, the global iterations are necessary. The maximum relative error 10^{-5} of the medium temperature is taken as the stopping criterion of iteration. The detailed procedure is as follows:

Step 1: Set the initial values of medium temperature.
Step 2: Solve Eq. (12) for the radiative intensity.
Step 3: Calculate the temperature distribution within the medium.
Step 4: Terminate the iteration process if the specified stopping criterion is satisfied. Otherwise, go to step 2.

3.1. Case 1: Linear refractive index and non-scattering medium

The finite element method is applied to a one-dimensional slab bounded by black walls. The refractive index of the medium within the slab varies linearly with the axis coordinate as $n(x) = 1.2 + 0.6x/L$. The medium within the slab is non-scattering. This case has also been used as a test case by Huang et al. [7] using the pseudo source adding method. The temperature distributions within the medium are presented in Fig. 2 for three values of slab optical thicknesses, namely $\tau_L = 0.01$, $\tau_L = 1.0$ and $\tau_L = 3.0$. As shown in Fig. 2, the FEM results are in good agreement with the results obtained by using the pseudo source adding method. In case 1, the number of iteration is less than six, and the time required for computation is less than 20 min on a personal computer with Intel Pentium Pro 450 MHz processor.

Table 1
Dimensionless radiative heat flux Ψ in the case of $n(x) = 1 + 2x/L$ and $\varepsilon_0 = \varepsilon_L = 1.0$

τ_L	$\Psi = 2\pi\int_{-1}^{1} I\mu d\mu / n_0^2\sigma(T_0^4 - T_L^4)$		
	RT [11]	MCDCRT [10]	FEM
0.1	0.9872	0.9882	0.9831
1.0	0.8720	0.8753	0.8701

Table 1 shows the dimensionless radiative heat flux in the case of $n(x) = 1 + 2x/L$ and $\varepsilon_0 = \varepsilon_L = 1.0$. By comparison with the results of the ray-tracing method (RT) developed by Lemonnier et al. [11] and Monte Carlo discrete curved ray-tracing method (MCDCRT) [10], it can be seen that FEM has good accuracy in solving the radiative transfer in one-dimensional semitransparent slab with variable spatial refractive index.

3.2. Case 2: Sinusoidal refractive index and non-scattering medium

In this case, nonlinear refractive index is studied. The refractive index of medium within the slab varies sinusoidaly with the axis coordinate as $n(x) = 1.8 - 0.6\sin(\pi x/L)$. The medium within the slab is non-scattering and the slab optical thickness is $\tau_L = 1.0$. Fig. 3 shows the temperature distributions within the medium for two different conditions of wall emissivity, namely $\varepsilon_0 = \varepsilon_L = 1$ and $\varepsilon_0 = \varepsilon_L = 0.7$. As shown in Fig. 3, the FEM results are in good agreement with the

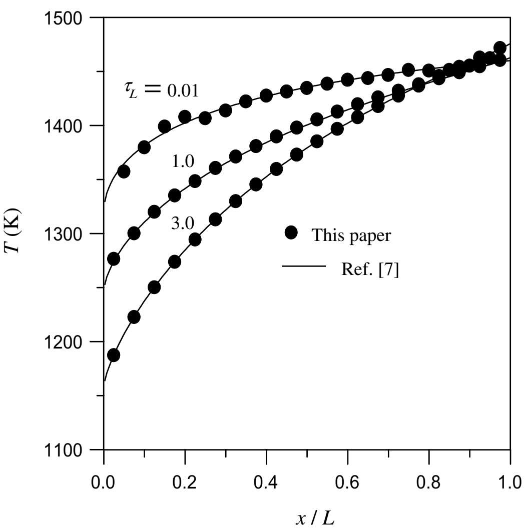

Fig. 2. Temperature distributions in the cases of $n(x) = 1.2 + 0.6x/L$, $\varepsilon_0 = \varepsilon_L = 1$ and $\omega = 0$.

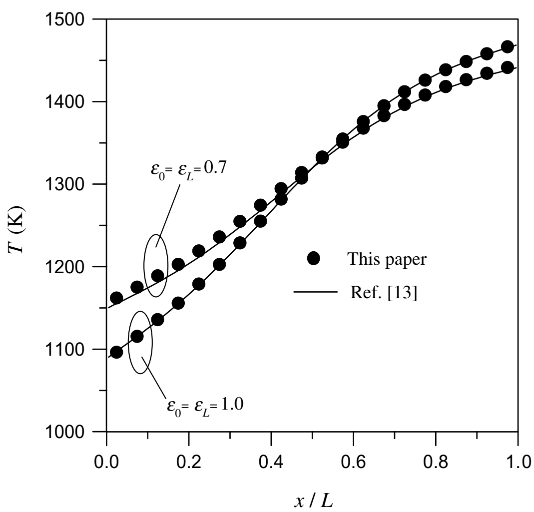

Fig. 3. Temperature distributions in the cases of $n(x) = 1.8 - 0.6\sin(\pi x/L)$, $\tau_L = 1$ and $\omega = 0$.

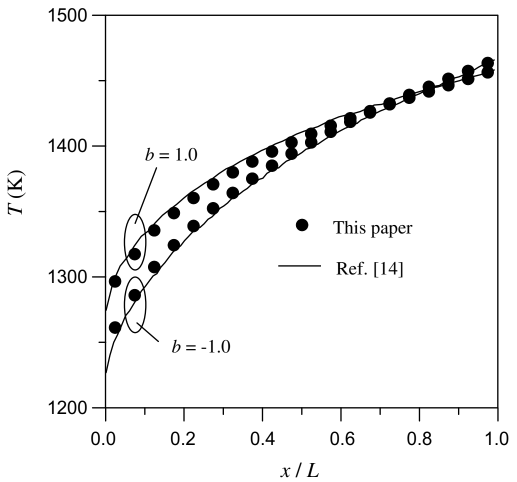

Fig. 4. Temperature distributions in the cases of $n(x) = 1.2 + 0.6x/L$, $\tau_L = 1$, $\varepsilon_0 = \varepsilon_L = 1$ and $\omega = 0.8$.

results obtained by using the pseudo source adding method [13]. The maximum relative error is less than 2%.

3.3. Case 3: Linear refractive index and anisotropically scattering medium

In this case, an anisotropically scattering medium is studied. The slab is bounded by black walls and the slab optical thickness is $\tau_L = 1.0$. The refractive index of medium within the slab varies linearly with the axis coordinate as $n(x) = 1.2 + 0.6x/L$. The single scattering albedo is $\omega = 0.8$, and the scattering phase function is assumed to be linear as $\Phi = 1 + b\mu\mu'$. Liu et al. [14] studied this case by Monte Carlo curved ray-tracing method. The temperature distributions within the slab are shown in Fig. 4 for two different values of asymmetry factor, namely $b = 1$ and $b = -1$, and compared to the results obtained from Monte Carlo curved ray-tracing method. The FEM results agree with those of Monte Carlo curved ray-tracing method very well. The maximum relative error is less than 3%.

4. Conclusions

To avoid the complicated computation of ray trajectories, a finite element formulation is developed to solve the radiative transfer in one-dimensional absorbing–emitting–scattering semitransparent slab with variable spatial refractive index. A problem of radiative equilibrium is taken as an example to verify this finite element formulation. The predicted temperature distributions are determined by the finite element method and compared with the data in references. The results show that the finite element formulation presented in this paper has good accuracy in solving the radiative transfer in one-dimensional absorbing–emitting–scattering semitransparent medium with variable spatial refractive index.

Acknowledgment

The support of this work by the National Nature Science Foundation of China (50336010) is gratefully acknowledged.

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